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PROGRESS REPORT.

Mr. V. G. Dalvi, B.A., (Bombay and Cantab)—Chief Secretary to the Indore Durbar, Indore,—has been elected a member of our Society.

2. As some numbers of the "American Mathematical Monthly" for 1913 were received last year, it was considered desirable to order the remaining numbers, and now the whole volume for 1913 is available for issue at the Library to members desiring to have it.

3. "*Matriculation Mechanics*"—by Briggs and Bryan, University Tutorial Press, 3rd edition, 1914, 3*s.* 6*d.*—is presented to our Library by the Publishers.

Calcutta University Calendar, Part III, 1914—has likewise been presented to the Library.

POONA,
30th September 1914.

D. D. KAPADIA,
Hony. Joint Secretary.

The Theories of Irrational Numbers, Part I:

Incommensurables, Convergent, and Earlier Geometrical Theories.

By Phillip E. B. Jourdain.

The aim of this historical and critical study is somewhat different from that of the other works that are known to us which deal with the development of the theory of convergence and allied topics. We shall, in fact, be concerned primarily with questions of principle; and the numerous and valuable deductions as to the criteria which enable us to decide whether such and such a series or class of series is convergent or not, which belong to technical mathematics, will only be shortly referred to.

The whole theory of convergence rests on one principle which is generally referred to as "Cauchy's general criterion," although the name of Bolzano might, perhaps, with greater propriety, be associated with this criterion. On the basis of this general criterion, various special criteria have been set up by Cauchy and many others for the purpose of deciding whether certain series do in fact converge, and a general theory of these special criteria has been the work of Dini, Paul du Bois-Reymond, and Pringsheim. But in all this work the validity of the general criterion itself was presupposed. This validity seemed, indeed, quite obvious to "intuition," and was either examined insufficiently or not at all, until there arose mathematicians who felt the need of tracing the logical connexion of mathematical theorems to the fundamental laws of arithmetic or even logic itself. Then it became evident that we cannot accept without a strict logical proof the theorem that, if a series fulfils such and such a condition, that series has a limit. From a logical point of view, the proposition that there is a limit is not in the very least proved by an appeal to those geometrical notions which seem to make it extremely plausible that the well-known geometrical interpretation of our conception of an arithmetical series points to something for which the arithmetical correspondent—if there be one—would be the limit in question. We must actually construct the entity we require out of our conceptions, and actually produce it for the mental vision of others. An appeal to "geometrical evidence," quite apart from its irrelevance, resolves itself into an appeal to faith, and the appeal is all the more vicious because, owing to certain psychological circumstances, it is almost universally responded to without reflection.

This first part is devoted, in the first place, to a sketch of the discovery of incommensurable magnitudes, the consequent severance, by the logically minded Greeks of ancient times, of geometry from arithmetic, and the gradual assumption of the exact correspondence between arithmetic and geometry, owing principally to the discovery of analytical geometry (§ I). In the second place, the rise of the careful treatment of the convergence of series is described. After a reference (§ II) to the work of Waring, Lagrange, and Gauss, which has been partially treated in other works on theh istory of mathematics, we proceed to a more detailed account of the contributions of Fourier (§ III), Cauchy (§ IV), and Bolzano (§ V) to the theory of the convergence of series. With Bolzano we have a very important indication of the fact that the existence of a limit to a series which fulfils a certain well-known condition must be proved, and the subsequent attempts of Hankel and Stolz to prove it are then dealt with (§ VI). Stolz's "proof" was founded on the conception of the least interval that includes all the limits to which a series that converges towards many limits can converge. This conception, which was known to Cauchy and Abel, had been forgotten, and was re-introduced by Paul du Bois-Reymond (§ VII). Du Bois-Reymond and Stolz apparently did not realize that the question of principle was the same as in the less complicated case of one limit, and it is instructive to study their errors and obscurities on this point. After a characterization (§ VIII) of the main purpose of theories of irrational numbers, the introduction of irrational numbers in De Morgan's and other text-books is examined (§ IX), and the geometrical theories (Guilmin, Bertrand) published before 1872 described (§ X).

§ I.

Pythagoras and the Pythagoreans held that number is the essence of all things, and that there is a complete parallelism between the conception of number and geometrical representation, in the sense that it is always possible to express the lengths of two straight lines as multiples of the same unit, provided that this unit was chosen small enough. However, Pythagoras himself discovered[1] and his discovery is one of the greatest of antiquity—that the diagonal of a square is incommensurable with its side; that is to say, no two numbers can express the ratio of these lines, so that the diagonal cannot be represented as a sum of points. The Pythagoreans seem to have believed that all geometrical entities could be so represented. This discovery was probably made in consequence of the discovery by Pythagoras of the relation between the

[1] Moritz Canto, *Vorlesungen über Geschichte der Mathematik*, Vol. I, 3rd, ed. Leipzig, 1907, pp. 181-184.

square on the hypotenuse and the squares on the other sides of a right-angled triangle, a theorem of which a particular case seems to have been known to the Babylonians[3] Chinese, and Egyptians, and which seems to have been arrived at in its generality, though probably merely by induction, by the Hindus.[2] If the sides of the triangle which include the right-angle between them are both of unit length, so that the hypotenuse is a diagonal of the unit square, it is obvious that the numerical measure of the length of this diagonal would be such that its square would be equal to 2. All attempts to find a fraction which should be the square root of 2 failed, and it would seem that at last the following proof of the incommensurability of the diagonal and the side was found.[3] Suppose that the diagonal were to the side as the integer d to the integer c, the ratio d/c being in its lowest terms. The Pythagorean theorem gives us the equation $d^2=2c^2$, so that we must conclude, that d is even and hence that c is odd. But, since d is even, we can write the theorem of Pythagoras as:

$$4(d/2)^2=2c^2 \text{ or } 2(d/2)^2=c^2,$$

which shows that c should be even. Hence the hypothesis that the ratio d/c expresses, in its lowest terms, the ratio of diagonal to side, must be given up.

At this point came the arguments of Zeno the Eleatic, which demonstrated to the Greeks the impossibility of making discontinuous plurality—the Pythagorean plurality of arithmetical points—account for the given continuous reality of space.[4] It would seem that Zeno—a supporter of the doctrine held by his master Parmenides that plurality is to be denied and that reality is unchanging—sought to show that motion is impossible. These arguments appear to have played a part in the necessity which the logically-minded Greeks felt for a rigid separation of arithmetic from geometry. These two sciences were only recombined after about eighteen hundred years of logical stagnation had dulled logical perception. But, perhaps, it was not this so much as it was a characteristic of the later and more western civilizations to pay less attention to logic while paying more to the observation of natural facts like motion.

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[2] *Cf.* Leon Brunschvicg, *Les étapes de la philosophie mathematique* Paris, 1912, p. 46.

[3] For the reasons for supposing that this proof (Euclid, *Elements*, Book X Prop. 117) dated from Pythagorean times, see Cantor, *op. cit.*, pp. 182-183 Brunschvicg, *op. cit.*, p. 47.

[4] Brunschvicg, *op. cit.*, p. 48; cf. pp. 154-155.

The Pythagorean discovery seems to have led, in spite of the Pythagoreans, to the rejection of the Pythagorean doctrine that numbers are fundamental to all reality, and to the adoption, by Plato, of a system in which geometry was fundamental.[5] Euclid[6] expressly said that incommensurable magnitudes are not related to one another in the same way as numbers. If, indeed, we represent the rational numbers such that their nth powers approximate to a given fraction which has no rational nth root, by points on a straight line, the sequence of points clusters round a point which does not correspond to any rational number.

The arithmeticians and algebraists of the middle ages and the Renascence—such as Leonardo of Pisa in his *Liber Abaci* of 1202—spoke of irrationals as "fictitious numbers" or "absurd numbers."[7] We find irrational "numbers" spoken of by Michæl Stifel in his *Arithmetica Integra* of 1544. He said that "an irrational number is not a true number," but this merely means that it is not a rational number. Stifel also remarked that every irrational number, just as every rational number, has a uniquely determined place in the series of numbers.

This is characteristic of the difference in the point of view of more modern mathematicians from that of the ancient Greeks. A tendency is observable to sacrifice the strict logic of the Greeks for the end of extending conceptions and methods into domains where it was felt—and perhaps only vaguely felt—that they applied. Thus the conception of a "variable" in mathematics was first introduced by the endeavour to include motion in space among the objects dealt with by mathematics. We know that the Greeks had often used the conception of motion in geometry as a means of discovery, but it was first in the work of Galileo that the conception in question began to be systematically used.

The discovery of analytical geometry by Fermat and Descartes and the discovery of the infinitesimal calculus by Leibnitz and Newton made it necessary to assume that every line could be measured in length by a number and thus practically identified geometry with the algebra of that time.[8] By the uncritical assumption of this correspondence bet-

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[5] *Ibid.*, pp. 45, 47, 48.

[6] *Elements*, Book X, Prop. 7. The fifth and tenth Books of the *Elements* are concerned, respectively, with the general theory of ratios and incommensurable magnitudes.

[7] From the latter name—"numeri surdi"—the English name of "surd" is derived.

[8] See A. Pringsheim *Encykl. der. math. Wiss.*, I.A. 3, pp. 49-53; and in greater detail, in the French edition (Pringsheim and Molk), I. 3, pp. 133-147.

ween numbers and lengths of lines, the distinction so clearly seen by the Greeks was slurred over and forgotton.

Newton, in his *Arithmetica Universalis* of 1707, understood by "number" a ratio of two quantities; and, up to within quite recent times, the introduction of irrational "number" was, explicitly or implicity geometrical.[9] We shall see later that Cauchy held the same opinion.

The conception of a "limit"[10] grew out of the "principle of exhaustion" used by the ancient Greeks, and the first to formulate the conception arithmetically was John Wallis. In his *Arithmetica Infinitorum* of 1655, he defined a number a to be a "limit" of a infinite sequence $a_0, a_1, a_2, \ldots$, if the difference $a-a_n$ becomes arbitrarily small when the integer n has a sufficiently great value. This definition, which is practically the same as that used at the present time, evidently presupposes the existence of a; but Cauchy was apparently the first to indicate the necessity of proving in an arithmetical way the existence of a limit. This he did in his proof (1823) of the existence of a definite integral of a continuous function. However, with Cauchy the question is only reduced to the assumption of the existence of a limit when a certain simpler criterion is fulfilled; and it is with this assumption that we are concerned in the present question of principle.

§ II.

The theory of convergence of series is principally due to mathematicians of the nineteenth century. Before this, only a few scattered theorems, due for the most part to Waring and contained in his *Meditationes analyticae* of 1781,[11] were known; and even Lagrange thought, in 1770,[12] that it is sufficient for the convergence of a series that its terms approach zero indefinitely. The introduction, by Lagrange in 1797, of a formula expressing the remainder after n terms of Taylor's series was not with a view to proving its convergence—for, this was a point upon which Lagrange did not touch—, but in order to be able to estimate the limit of error when we break off with a finite number of terms.

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[9] See *Ibid.*, pp. 153-154.

[10] *Cf.* Pringsheim, *Encykl. der. math. Wiss.*, I.A. 3, pp. 63-65.

[11] *Cf.* M. Cantor, *Vorlesungen uber Geschichte der Mathematik*, Vol. IV., Leipzig, 1908, pp. 275-276, 285-287 (article by E. Netto). On the history of convergence, see, besides, the third and fourth volumes of Cantor's *Geschichte*, R. Reiff, *Geschichte der unendlichen Reihen*, Tubingen, 1889; A. Pringsheim, *Encykl. der math. Wiss.*, I A. 3, 1898, pp. 78-79, and the corresponding part of the French edition. Reiff did not mention Waring.

[12] Reiff, *op. cit.*, p. 148.

Still, Lagrange's *Theorie des Fonctions* of 1797 may be said [13] to form the passage from the old to the modern treatment of infinite series, and Lagrange's investigations on the remainder of Taylor's series appear to have influenced Fourier and, partly through him, Cauchy, to evolve a strict theory of convergence.

Gauss, in his inaugural dissertation of 1799, said, in his criticism of a procedure of d'Alembert's for finding the form of the roots of an algebraic equation:[14] "Infinitely small magnitudes were used by d'Alembert in a looser way than is compatible with mathematical rigour, or than a far-seeing mathematician would allow at the present time, when those magnitudes are rightly regarded with mistrust. Also the leap from an infinitely small to a finite value of Ω [an infinitely small value of the rational and whole function considered] is not clear enough. The assertion that Ω can receive any finite value cannot be concluded from the possibility of an infinitely small value of Ω. This assertion follows from the fact that, when Ω is small enough, the approximation to the true value of ω [the value of x corresponding to Ω] increases with the number of the terms [of the series for ω in terms of Ω) retained, because of the strong convergence of the series. In other words, the equation which gives the relation between ω and Ω (or between x and X) is satisfied so much the more nearly as we take more terms to get ω. But this whole method of drawing conclusions seems too indefinite for any rigorous inference to be drawn from it. Besides this, I may also remark that there are series which always diverge however small the value of the magnitude in powers of which the development is effected, so that if we proceed far enough, we arrive at terms which are greater than any given magnitude. This happens if the coefficients of the series form a hypergeometric series."

Gauss added a note remarking, by the way, that there are series "which first converge very strongly and then more and more weakly, and finally diverge more and more; but, in spite of this, give the sum almost exactly if not too many terms are taken." He mentioned that this fact did not seem to have been noticed before, and that it was desirable to show the reason of this behaviour and to determine in how far such a sum might be securely taken as correct. Again, Gauss remarked[15] that "from the supposition that X can take the value S but not the value U does not follow that between S and U a value T must lie which X can reach but not surpass. Here a case is overlooked: it is

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[13] *Ibid*, pp. 153, 159.

[14] *Ostwald's Klassiker*, No. 14, p. 11.

[15] *Ibid.*, p. 12.

possible that between S and U there lies a limit to which U can approach arbitrarily near without reaching. From the grounds advanced by d'Alembert only follow that X can surpass every value which it reaches by a finite magnitude. If therefore it is, say S, it can be increased by a finite magnititude Ω: then we can have a new increase Ω', and then a further increase Ω'' and so on. But there is no last increase. However many increments have been added, a new one can always be added. But, though the number of the possible increments is infinite, it may happen that, if Ω', Ω'',..... continually decrease, the sum

$$S+\Omega+\Omega'+\Omega''+\ldots$$

never reaches a certain limit, however many terms may be taken. This case cannot indeed occur if X is a whole algebraic function of x, but it must be proved that the case in question cannot happen...".

We shall refer to this passage again when we come to consider Bolzano's formulation of the conception of "upper limit." At present, it merely serves the purpose of showing that the difficulties which were brought forward in the first two paradoxes of Zeno about motion, and which are only solved by the ability to conceive an infinite series converging to a finite limit, were most decidedly overcome by Gauss.

Gauss, in his memoir on the hypergeometric series, published in 1812, used the principle of comparison of a series with the geometric series, which is known to have a finite and assigned limit under a certain simple condition[16]. This principle requires for its rigorous foundation one of the modern theories of irrational number. Of course, we cannot think it probable that Gauss had penetrated so deeply into the foundations of analysis as to have evolved a sound theory of irrational numbers, but it is certain that he had formed the idea that it is necessary to assure oneself that the sum of the terms of a series approach a finite limit before we can calculate with this limit—"the sum."

§ III.

In 1811, Fourier[17] gave a definition of convergence which showed that he had correct and fairly precise ideas on the conditions under which "the sum" of an infinite series could be calculated with Fourier's work was not actually printed until 1822, and it is impossible to say how far, if at all, Cauchy's conception of convergence, which was published in 1821, was indebted to that of Fourier.

[16] Reiff, *op. cit.*, p. 162. Gauss (see *ibid.*, pp. 163-166) actually continued the investigation of convergence so far as to decide the convergence or otherwise of the series considered by him, in every case.

[17] Fourier presented a memoir on the theory of the conduction of heat to the French Institute in 1807 (on it cf. below and *Oeuvres de Fourier*, Vol. I, p. iv ;

Speaking of a certain trigonometrical series in his *Theorie analytique de la Chaleur* of 1822, Fourier said: "It would be easy to prove that this series is always convergent, that is to say that writing, instead of y, any number whatever, and following the calculation of the coefficients, we approach more and more to a fixed value; so that the difference of this value from the sum of the calculated terms becomes less than any assignable magnitude. Without stopping for a proof,

Vol. II, pp. vii, xii, 215-221). For 1811, the French Academy proposed for a prize the finding of a mathematical theory of the propagation of heat, and a long essay by Fourier, which was deposited in the archives on Sept. 28th 1811, was awarded the prize in 1812 (see. *Ibid*, Vol I, pp. vii-viii). When, after Delambre's death, Fourier became perpetual secretary, he had the memoir printed without any alteration, in the *Memoires de l'Acadèmie Royale des Sciences de l'Institute d France* (vol. IV, 1819 and 1820, published 1824, pp. 185-556, and Vol. V, 1821 and 1822, published 1826, pp, 153-246), after the publication (1822) of his *Theorie analytique de la Chaleur* (we shall also refer to the English translated by A. Freeman cited below). Fourier doubtless wished to establish thus, in an incontestible manner, his rights of priority, for the first part of the memoir of 1811 (that which appeared in the volume for 1819-20) only differs in quite secondary points from the definitive version in the *Theorie de la Chaleur*. Thus Darboux did not reproduce the first part; but, as the second, printed in the *Memoires* for 1821-2, is of greatest interest, it is reproduced in the *Oeuvres de Fourier*, Vol. II, pp. 1-94.

Here we have to consider Fourier's precise conception (cf. Pringsheim, *Encykl. der math. Wiss.*, I. A. 3, p. 146, where the reference is wrong) of convergence, on p. 315, pages 313-315 are exactly reproduced in article 228 of the *Théorie* (*Oeuvres*, Vol. I, pp. 221-222; *Freeman* pp. 195-197,—see below). A careful comparison of Fourier's memoir of 1811 with the *Théorie* has also shown:

(1) That the word "convergent is used on p. 269 of the *Mémoires* (which corresponds to article 177 of the *Théorie*), but the explanation of it, which is given in the *Théorie*, is not given in the *Mémoires*;

(2) Article 178 of the *Théorie* is exactly given on p. 270 of the *Mèmoires*;

(3) Articles 184-186 of the *Théorie* are given, in essentials, on pp. 275-278 of the *Mémoires*;

(4) Divergent series are used for calculation on, for example, p. 262 of the *Mémoires*, which exactly corresponds to article 171 of the *Théorie* (*Oeuvres*, Vol. I., p. 149);

(5) An oscillating series is used for calculation (in fact, for $1-1+1-\ldots$ is put $\frac{1}{2}$) in the *Memoires*, as in article 218 of *Theoric* (see *Oeuvres*, Vol. I, p. 206 *Freeman*, p. 182). But the calculation with an oscillating series in article 235 of the *Theoric* (see *Oeuvres*, Vol. I, p. 234; *Freeman* p. 288) is not in the *Memoires*;

(6) The Geometrical construction given by Fourier for proving the convergence of his integrals and series (cf. the text below) were intentionally omitted from the memoir of 1811. Fourier's modern conception of convergence certainly dated from before 1811, but no statement dating from an earlier year has been hitherto published.

which the reader may supply..."[18]. Again, convergence, says Fourier.[19] 'does not result solely from the fact that the values of the terms diminish continually; for, this condition is not sufficient to establish the convergence of a series. It is necessary that the values at which we arrive on increasing continually the number of terms should approach more and more a fixed limit, and should differ from it only by a quantity which becomes less than any given magnitude: this limit is the value of the series."

In article 178 of the *Theorie*,[20] Fourier gave an indication of the way in which the curve represented by a finite number of terms of the series

$$\cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \ldots$$

behaves as this number is increased; and, in the immediately following third section of the third chapter,[21] Fourier summed the first m terms of the above series, regarded this sum as a function (y) of x and m, and by a differentiation, certain other operations, integration by parts, expressed y as an infinite series arranged according to negative powers of m. Then it appears that, the more m increases, the more y approaches a constant value independent of x. Darboux[22] remarked on this passage that, in the study of some particular series, Fourier followed exactly the same method, which Dirichlet later used, to provide, for the first time, a rigorous theory of trigonometrical series.[23]

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18 *Theorie*, art. 177; *Oeuvres de Fourier*, Vol. I, p. 156; English translation, with notes by R.L. Ellis and Freeman, by Alexander Freeman (*The Analytical Theory of Heat*, Cambridge 1878; referred to here as *Freeman*), pp. 143-144. A passage on p. 433 (art. 418; *Oeuvres*, Vol. I. p. 503) is to the same effect; while a passage on p. 207 (art. 235; *Oeuvree*, Vol. I, p. 232), which runs: "The series arranged according to sines or cosines of multiple arcs are always convergent; that is to say, on giving to the variable any value whatever that is not imaginary, the sum of the terms converges more and more to a single fixed limit, which is the value of the developed function," is of interest as showing that the series which Fourier considered only converge along the real axis (cf. P. du Bois-Reymond's paper of 1876 reprinted in No. 186 of *Ostwald's Klassiker*, pp. 6-8).

19 *Theorie*, art. 228; *Oeuvres*, Vol. I, pp. 221-22; *Freeman*, pp. 196-197.

20 *Oeuvres*, Vol. I, pp. 157—158; *Freeman*, p. 144. In the *Memoires*, this passage occurs on pp. 269—270.

21 "Remarques sur ces séries", *Oeuvres*, Vol. I, pp, 158—169; *Freeman*, pp. 145—154. These investigations are on pp. 270—280 of the *Memoires*.

22 *Oeuvres de Fourier*, Vol. I, p. 158 *n*.

23 This method is to express by a definite integral the sum of the m first terms of the series and then to seek the limit of this integral.

Fourier, after treating other analogous series in a like manner, proceeded[24] to determine the limits between which the remainder of such a trigonometrical series lies.

In his first investigations, which were presented to the Institute of France in 1807, or in the extensive notes, which were partly concerned with the convergence of series and which Fourier added in 1808 or 1809 to the original memoir[25] to reply to or forestall the objections of Lagrange and Laplace,[26] Fourier gave some geometrical constructions for demonstrating the convergence of his double integrals and his series, which were left out of the memoir of 1811.[27]

That this geometrical argument of Fourier's[28], although it is not carried through rigorously, contains the germs of Dirichlet's celebrated investigation, was remarked by Reiff[29] and Darboux.[30]

§ IV.

Before the time of Weierstrass, the introduction of irrational numbers was, explicitly or implicitly, geometrical. We will first examine Cauchy's *Cours d'Analyse* of 1821.[31]

In the "Préliminaires", Cauchy explicitly said[32] that the word "number" was to be understood to denote the entities which arise from the measurement of magnitudes. Again, he said[33] "Just as the idea of number arises from the measurement of magnitudes, so that of *quantity* arises when we consider each magnitude of a given species as serving for the increment or diminution of another fixed magnitude of the same species. We will agree to range absolute numbers, which are not preceded by any sign [denoted by capital letters], in the class of positive quantities." Relative numbers (numbers with sign or "quanti-

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24 *Théorie*, arts. 185-188; *Oeuvres*, Vol. I, pp. 165-168; *Freeman*, pp. 150-153. This investigation is given in the *Memoires* pp. 275-280.

25 *Théorie*, "Discours préliminaire"; *Oeuvres*, Vol. I, p. xxvi; *Freeman*, p. 9.

26 *Oeuvaes de Fourier*, Vol. II, p. vii.

27 *Thorie*, "Discours" and art. 415; *Oeuvres*, Vol. I, pp. xxvi., 404; *Freeman*, pp. 10, 426.

28 *Theorie*, art. 423; *Freeman*, pp. 438-440; *Oeuvres*, Vol. I, pp. 509-513.

29 R. Reiff, *op. cit.*, pp. 184-186.

30 *Oeuvres de Fourier*, Vol. I, pp. 499, 511-514; cf. the note on p. 158 referred to above.

31 *Cours d'Analyse de l' le royale Polytechnique*: Première Partie [no further parts appeared], *Analyse algebrique*, Paris, 1821, Chap. VI, *Oeuvres* (2), Vol. III, **Paris**, 1897.

32 *Ibid*, p. 17.

33 *Ibid*, p. 333.

ties ") were denoted by small italicized letters; and Cauchy[34] remarked that we must carefully distinguish between operations relating to numbers and those relating to quantities.

Near the beginning of the book, Cauchy defined a "limit" as follows: "When the successive values attributed to a variable approach indefinitely a fixed value, so as to end by differing from it as little as wished, this last is called the 'limit' of all the others"; and remarked that "thus an irrational number is the limit of the various fractions which furnish more and more approximate values of it." If we consider—as, however, Cauchy did not do, although many others have done—the latter statement as a definition, so that an 'irrational' number is defined to be the limit of certain sums of rational numbers, we presuppose that this sum *has* a limit. In a note at the end of the book, Cauchy[35] explained the multiplication of the number A by the number B, where B is irrational: "...We can obtain, in terms of rational numbers, values which are more and more approximate to B. We easily show that, under the same hypothesis, the product of A by the rational numbers in question approximates more and more to a certain limit. This limit will be the product of A by B." And the explanation of A^B, when B is an irrational number, is[36] on the same principles.

Thus, Cauchy did not define real numbers arithmetically, but presupposed their existence, by assuming a correspondence between the system of real numbers and the system of points on a straight line.[37]

Towards the beginning of the fifth chapter of Cauchy's work, he[38] got, for instance, from the functional equation between continuous functions

$$\phi(x+y)=\phi(x)+\phi(y),$$

the equation $\phi(x)=x\phi(1)$, by proving the latter for rational x's, then varying x so as to approach an irrational limit, and then "passing to the limit." The legitimacy of this passage depends on the theorem:[39] If $x, y, \ldots$ converge respectively towards the limits $X, Y, \ldots$; and $f(x, y, \ldots)$ is continuous with respect to each variable in the neighbourhood of the system of values $x=X$, $y=Y, \ldots f(x, y, \ldots)$ will have $f(X, Y, \ldots)$ as limit. Cauchy's method of conclusion by "passing to the limit" may also be illustrated by his argument, at the end of § 1 of the sixth chapter of

[34] *Ibid*, p. 335.

[35] *Ibid.*, p. 337.

[36] *Ibid.*, pp. 341-342.

[37] For a critique of this note of Cauchy's, see H. Hankel, *Theorie der complexen Zahlensysteme*, Leipzig, 1867, p. 14.

[38] *Op. cit.*, p. 99; cf. pp. 102, 108-109, 111.

[39] *Ibid.*, p. 47.

his *Cours*, about the continuity of the sum of a series of functions. If u_v is a continuous function of x in the neighbourhood of $x=x_0$, s_n r_n $(=s-s_n)$, and s are functions of x, of which s_n is evidently continuous in the neighbourhood of x_0; "then consider the increments which these three functions receive when x increases by an infinitely small quantity α. The increment of s_n will be, for all possible values of n, an indefinitely small quantity; and that of r_n will become insensible at the same time as r_n, if a very considerable value is attributed to n. Consequently the increment of the function s can only be an indefinitely small quantity," and so s is continuous in the neighbourhood of $x=x_0$. This theorem was applied, in § IV of the same chapter, to prove that the sum of a convergent power-series is a continuous function, and this was applied to a proof of the binomial theorem. At this point we arrive at the considerations of Abel, Weierstrass, Stokes, Seidel and Cauchy; but we shall not pursue the subject in this place.[40]

Cauchy gave the same definition of "convergence" as Fourier. Where

$$s_n=u_0+u_1+\ldots+u_{n-1}$$

"if", said Cauchy[41], "for values of n which always increase, the sum s_n approaches indefinitely a certain limit s, the series will be called convergent, and the limit in question will be called the sum of the series. On the other hand, if, while n increases indefinitely, the sum s_n does not approach any fixed limit, the series will be divergent and will no longer have a sum."[42] Cauchy then illustrated this distinction by the

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40 Cf. *Archiv der Math. und Phys.* (3), Vol. XIV, pp. 300-301

41 *Oeuvres* (2), Vol. III, pp. 114-115; cf. Reiff, *op. cit.*, p. 167.

42 We collect here a few typical instances of the same conception—which depends on the existence of a limit—of *convergence*:

De Morgan (*Elements of Algebra*, 2nd ed., London, 1837, p. 179) defined a "convergent" series as one in which a limit towards which we approximate by continually adding more and more of its terms exists, considered (pp. 159-162) the geometrical series, which has a rational limit, derived (p. 182) the criterion of convergence lim. u_{n+1}/u_n 1, and proved by this criterion that the exponential series is convergent. The result that the exponential series, even though it does not tend to a rational limit, is convergent, is a consequence of the tacit assumption, made in the derivation of the criterion, that a series, every term of which is numerically less than the corresponding term of a convergent series is convergent.

De Morgan said (p. 95): "Every number and fraction has a root of every order, either exact or approximate;" and cautioned (p. 101): "Whenever any new process is introduced, or any new expression, it must be proved, and assumed, that problems involving that process admit *quam proxime*, if not of exact, solutions."

The same doctrine of convergence is also given in De Morgan's *Elementary Illustrations of the Differential and Integral Calculus*, published at London in 1832

geometrical progression 1, x, x^2,..., which converges if $x<1$ and diverges if $x>1$, and proceeded: "By the above principles, in order that the series

$$u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$$

may be convergent it is necessary and sufficient that increasing values of n make the sum s_n converge indefinitely towards a fixed limit s; in other words, it is necessary and sufficient that, for infinitely great values of n, the sums s_n, s_{n+1}, s_{n+2},...differ from the limit s, and consequently from one another, by indefinitely small quantities. Now, we have

$$s_{n+1}-s_n=u_n, \; s_{n+2}-s_n=u_n+u_{n+1}, \; etc \;;$$

thus it is necessary that the general term u_n should decrease indefinitely as n increases. But this condition does not suffice; the various sums

$$u_n+u_{n+1}, \; u_n+u_{n+1}+u_{n+2} \ldots,$$

taken in any number, must end by constantly having values which are numerically less than any assignable magnitude. Reciprocally, when these various conditions are fulfilled, the convergence of the series is assured." Then Cauchy showed that the harmonic series is not convergent although the general term decreases indefinitely. Further, the series

$$1, \frac{1}{1}, \frac{1}{1\cdot 2}, \ldots, \frac{1}{1\cdot 2 \cdots n}, \ldots$$

is convergent, said Cauchy, because each term is less than the corresponding term of a convergent geometrical progression.

This general criterion of convergence occupies a very prominent position in theories of irrational numbers. It cannot, however, be practically used with many series, and accordingly Cauchy gave a series of special criteria, based on the principle of comparison with series known to converge. Here we must pass over the development of the theory of convergence-criteria in the hands of Cauchy and others,[43] but there is one point about the distinction of two kinds of criteria and their efficacy that must be touched upon, as it is very closely connected with the principles of the theory of irrational numbers.

Cauchy considered, in the second section[44] of Chapter VI., series all of whose terms are positive, and obtained, by comparison with the convergent geometrical series, the following criterion of convergence of the series u_0, u_1, u_2,... "Find the limit or limits[45] to which $(u_n)^{1/n}$ con-

(new edition, Chicago 1909, pp. 17-19), and bound up with his large *Differential and Intergral Calculus*, London, 1842. See also pp. 118, 222, 234, of the last-named work.

Nearly all writers follow Cauchy in making the meaning of "convergence" imply the existence of a limit. Cf. *Encykl. der math. Wiss.*, Vol. I, p. 77.

[43] Cf. Reiff, *op. cit.*, pp. 167sqq.; Pringsheim, *Encykl. der math. Wiss.* Vol. I., A. 3, p. 189.

[44] *Oeuvres* (2), Vol. III., pp. 121-128.

[45] For the case of the occurrence of many limits, cf. §VII below.

verges when n is infinite, and let k be the greatest of these limits (or, in other words, the limit of the greatest value of the expression in question). The series converges if $k<1$ and diverges if $k>1$." After the proof of this theorem, Cauchy proceeded: "In a great number of cases, we can determine the value of k by theorem IV of section III of Chapter II. Thus, whenever u_{n+1}/u_n converges to a limit, this limit will be the value of k. From this, Cauchy got a theorem in which a more convenient but less general criterion was stated.

Cauchy[46] emphasized the advantage of the first criterion over the second, for, the upper limit in the first always exists, while that in the second, may not. However, this important distinction has often been overlooked by mathematicians.[47] Nowadays, we follow du Bois-Reymond[48] in distinguishing criteria of "the first" and "second kind."

We have seen that Cauchy defined a "convergent" series as one which has a limit, and then stated a necessary and sufficient condition for convergence, in which no existence of limits was assumed. The necessity of this condition—the proof that if one and only one limit s exists, then, where ϵ is an arbitrarily small positive non-zero number, which is of course rational unless we have already defined the system of real numbers, a whole number n can be found such that

$$| s_n - s_{n+m} | < \epsilon,$$

whatever interger m is—is easily shown. The sufficiency of the condition—that the last inequality implies the existence of a limit—was asserted, but not proved, by Cauchy. Cauchy's theory seems, in fact, to have been essentially of a geometrical nature.

Before Cauchy, Bernard Bolzano (1817) published a memoir[49] in the course of which he noticed that this criterion stated an important property of series, and went beyond Cauchy in attempting to prove the sufficiency of this criterion for establishing the existence of a limit. This proof failed, and the existence of a limit was, after all, only assumed as a result of an appeal to those of us who "have a correct conception of magnitude." But, at any rate, that point was emphasized which seems to have been universally overlooked by writers of text-books up to quite modern times. It was this oversight which led many persons into the vicious circle of defining a convergent series as one for which a limit exists, and then supposing that certain series defined, by approximation, certain 'irrational' numbers not hitherto considered.

(*To be continued.*)

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46 *Cf.* Pringsheim, *loc. cit.*, p. 81.
47 *Ibid.* Reiff also did not notice this.
48 *Journ. für Math.*, Vol. LXXVI, 1873, p. 61. *Cf.* Pringsheim *loc it.c.*,, pp. 80-90
49 Reprinted, with notes, in No. 153 of *Ostwald's Klassiker*.

Isogonal Transformation.

[The substance of this paper was read at a meeting of the Presidency College Mathematical Association. References: Salmon's *Conic Sections*, §§ 55, 297 ; Salmon's *Higher Plane Curves*, §§ 283, 34-36.]

1. If S, H be two points in the plane of a triangle ABC such that

angle BAS = angle HAC
„ CBS = „ HBA
„ ACS = „ HCB,

then the points S, H are said to be isogonally conjugate with respect to ABC, and their trilinear co-ordinates with respect to ABC are reciprocal.

If each point of a figure is replaced by its isogonal conjugate, a new figure is evidently formed; the transformation so effected is called *isogonal transformation.*

The only *self-conjugate* points are the *in* and *ex* centres of ABC.

The conjugate of any point on a side of ABC is the opposite vertex. A vertex as such, has therefore no *definite* conjugate point, but regarded as lying on a continuous curve its conjugate is a definite point on the opposite side, *viz.*, its intersection with the isogonal conjugate of the tangent at the vertex.

The conjugate of any point on the circumcircle of ABC is at infinity, and the circular points at infinity (being common to the circumcircle and the line at infinity) are conjugate points.

2. The following properties are easily established :—

(i) The transform of a *continuous* curve is a continuous curve.

(ii) Each side of the triangle of reference cuts a curve of the nth degree in n points, and therefore the transform passes through the opposite vertex n times, that is, has a multiple point of the nth order at the vertex.

(iii) The transform of a curve of the nth degree is in general a curve of the $2n^{th}$ degree.

For, if $f(\alpha, \beta, \gamma)=0$ be a curve in trilinear co-ordinates, the transform is seen to be $f(\beta\gamma, \gamma\alpha, \alpha\beta)=0$.

If, however, the original curve has a multiple point of the kth order at A, its equation may be put in the form

$$\alpha^{n-k}\phi_k+\alpha^{n-k-1}\phi_{k+1}+\ldots\ldots\ldots\phi_n=0$$

where ϕ_r is a homogeneous function in β, γ of degree r.

The transform is therefore obtained by replacing α, β, γ by their reciprocals and multiplying by $(\alpha^{n-k}\beta^n\gamma^n)$. It is thus of degree $(2n-k)$.

Hence the degree of the transform is lowered by k, corresponding to a multiple point of the kth order at a vertex on the original curve.

Note.—Though the degree of a curve is generally *doubled* by isogonal transformation, the transform has multiple points of the nth order at each vertex of the triangle of reference, and therefore (by the above rule) a second transformation will give a curve of degree $(2 \cdot 2n - n - n - n) = n$, and is in fact the original curve.

3. *Applications* :—

(1) *From any point two tangents can be drawn to a conic.*

Transforming with respect to a triangle inscribed in the conic, we deduce that *two conics can be described through four points to touch a given straight line.*

(2) *If a triangle circumscribes a parabola its focus lies on the circumcircle of the triangle.*

For, the foci of an inscribed conic are conjugate points ; and since one of them is at infinity, the other must lie on the circumcircle.

(3) *If a conic touch the sides of a triangle and its foci lie on a rectangular hyperbola through the vertices of the triangle, the major axis of the conic passes through the circumcentre.* [Mathl. Tripos.]

Since, the circumscribed rectangular hyperbola also passes though the orthocentre, its isogonal transformation is a straight line passing through the circumcentre. Also, the two foci which lie on the rect. hyperbola are themselves conjugate to each other. Hence, the transformed straight line passes through these foci. In other words, the major axis of the in-conic passes through the circumcentre.

(4) *A conic touches four given straight lines. Show that the locus of the foci is a cubic curve passing through the six intersections of the given lines.*

In the first place, it is easily seen that the six intersections are special positions of the foci of the inconic. For, each diagonal of the quadrilateral is a line conic belonging to the system of inscribed conics whose foci are its extremities.

Secondly, the two foci must trace the same locus. For, there is no intrinsic difference in the relation of either focus to the quadrilateral.

Thirdly, the locus cannot be a nodal curve. For, then, there would be more than one conic of the system having the node for a focus.

Hence, transforming with respect to the triangle formed by any three of the given straight lines, the locus is its own conjugate, so that, by § 2 (iii),

$$2n-3=n, \text{ or } n=3,$$

n being the degree of the locus.

4. *Cross-Ratio Properties* :

(1) If (PQRS) be a set of points whose isogonal transformation is (P′Q′R′S′), then the cross-ratios of the pencils A(PQRS), A(P′Q′R′S′) are equal, where A is a vertex of the triangle of reference.

The proof is obvious, since the cross-ratio of a pencil is a function of the sines of the angles between the different rays of the pencil, and isogonal transformation does not affect the angle between the rays joining two points to a vertex of the triangle of reference.

(2) If (P, Q, R, S) be points on a circumconic, their transform (P′Q′R′S′) is a range on a straight line. Thus

$$A\,(PQRS)=A\,(P'Q'R'S') \text{ by (1).}$$
$$=\quad (P'Q'R'S')=B\,(P'Q'R'S')$$
$$=B\,(PQRS) \text{ by (1).}$$

Hence we have the constant cross-ratio property of conics. We also learn from the above that *a range on a straight line transforms into an equi-cross range on a conic.*

(3) "*A four-point system of conics cuts a straight line in an involution range.*"

Transform, taking three of the four points to form the triangle of reference, and we have

"*Lines through a fixed point cut a conic in corresponding point of an involution range.*"

(4) The pole and polar property of conics may be stated thus :

"'*Tangents at corresponding points of an involution range on a conic et on a straight line.*"

Transforming with respect to an inscribed triangle this becomes:

"*If a three-point system of conics touches a straight line at points of an involution range, the intersection of corresponding members lies on a conic through the three points.*"

As a particular case we may notice the following :

"*Parabolas circumscribing a triangle and having their axes at right angles intersect on a conic.*"

(5) From the involution property of conics follows an interesting involution propetry of nodal cubics.

If a conic passes through the vertex A and cuts the base BC in points distinct from B, C, its isogonal transformation is a nodal cubic having the node at A, and passing through B and C. Now, if transversals $(B\,P_1\,Q_1)$, $(B\,P_2\,Q_2)$...are drawn through B to cut the conic in (P_1Q_1), $(P_2\,Q_2)$,......then $A(P_1\,P_2......Q_1\;\;Q_2\,..)$ is an involution pencil. Transforming, we find that *in the nodal cubic lines through any point B cut the cubic again in pairs of points* $(P_1'Q_1')$ $(P_2'Q_2')$*...such that* $A(P_1'P_2'......Q_1'Q_2'...)$ *is a pencil in involution, where A is the node.*

A. Narasinga Rao.

Note on the Diameter of a Quadrilateral.

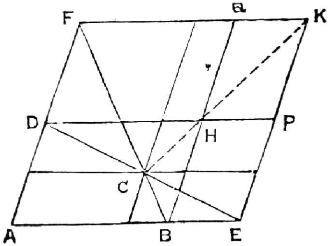
The collinear property of the midpoints of the diagonals of a complete quadrilateral has been proved in Text Books in various ways [*Vide:* (1) Casey: *Sequel to Euclid*, p. 5. (2) Richardson and Ramsey: *Modern Plane Geometry*, p. 72. (3) Askwith: *Pure Geometry*, p. **151**. (4) Russell: *Pure Geometry*, pp. 34, 241. (5) Durell: *Plane Geometry*, Pt. I, pp. 118, 119.]

The following elementary proof may be of interest.

Let ABCD be a quadrilateral of which AC, BD, EF are the diagonals. Complete the parallelograms AH, AK as in the figure, and draw parallels through C to AB, AD.

Now in the parallelogram AQ, the complements CA, CQ are equal; and in the parallelogram AP, the complements CA, CP are equal.

$\therefore$ $\quad$ ▭^{m} CP = ▭^{m} CQ.



$\therefore$ $\quad$ CHK is a straight line.

Hence, also, the middle points of AC, AH, AK are in a straight line. But the middle points of AH, AK are the same as the middle points of BD, EF. Therefore the middle points of the diagonals AC, BD, EF are collinear.

M. T. NARANIENGAR.

The Face of the Sky for November and December 1914.

The Sun

enters the winter Solstice on December 22 at 9·54 P.M.

Phases of the Moon.

	November.			*December.*		
	D.	H.	M.	D.	H.	M.
Full Moon ...	3	5	19	2	11	51
Last Quarter ...	11	5	7	10	5	2
New Moon ...	17	9	32	17	8	5
First Quarter ...	24	7	9	24	1	55

Planets.

Mercury is stationary on November 16 and attains its greatest elongation (19°51′ W) on November 24.

Venus is still an evening star, but approaching the Sun with which it is in inferior conjunction on November 27. It is in conjunction with the Moon on December 15 with Mars on November 22. During these months it is near the boundary between Scorpio and Libra.

Mars is in conjunction with the Sun on December 24 and with the Moon on November 18 and December 17. The planet is in Sagittarius in these months.

Jupiter is in quadrature to the Sun on November 7 and is in conjunction with the Moon on December 21. In these months it is near the boundary between Capricornus and Aquarius.

Saturn is in Taurus. It is in conjunction with the Moon on December 4 and December 31 at 6·51 P.M.

Uranus is in conjunction with the Moon on December 20.

Neptune is stationary on November 3 and is in conjunction with the Moon on November 9 and December 7.

V. RAMESAM.

SOLUTIONS.

Question 339.

(M. BHIMASENA RAU) :—The incircle of a triangle ABC touches AC, AB in E, F, and E′, F′ are the midpoints of AC, AB. Show that the radical axis of the circles described on EF′, E′F as diameters passes through the in-Fuerbach point.

Solution by N. Sankara Aiyar, M. A.

Take the medial triangle D′E′F′ as the triangle of reference. Then the equation of the *n. p. c.* of ABC is

$$S\equiv a\beta\gamma+b\gamma\alpha+c\alpha\beta=0.$$

Now the equation of BC is $b\beta+c\gamma=0$.

Hence the equation of the incircle of ABC is

$$\sqrt{(s-a)(b\beta+c\gamma)}\pm\sqrt{(s-b)(c\gamma+a\alpha)}+\sqrt{(s-c)(a\alpha+b\beta)}=0.$$

The common point of these two circles is easily seen to be

$$\left(\frac{1}{b-c},\ \frac{1}{c-a},\ \frac{1}{a-b}.\right)$$

The point E is found to be $c(a-c)$, ac, $a(c-a)$.

Similarly F is found to be $b(a-b)$, $a(b-a)$, ab.

If P (l,m,n) be a point on the circle on EF′ as diameter, then PE and PF′ are ⊥r. That is $m\alpha-l\beta=0$, and $\alpha a(mc-ma-cn)+\beta(a-c)(cn+ab)+c\gamma(al-ma+c)=0$, should be perpendicular.

Thus $am(mc-ma-cn)-l(a-c)(cn+ub)+b\cos A\cdot c(al-ma+c)$ $mc\cos B\,(al-ma+c)-\cos_{|}c\,\{m(a-c)(cn+al)-al(mc-ma-cn)\}=0.$

Simplifying, we get for the circle on EF′ as diameter

$$c\,S+(a\alpha+b\beta+c\gamma)\,\{\beta(a-c)\cos C+\alpha(a-c-c\cos A)\}=0.$$

Similarly for the circle on E′F as diameter.

Hence the radical axis of the two circles is

$$\frac{\beta(a-c)\cos C+\alpha(a-c-c\cos A)}{c}=\frac{\gamma(a-b)\cos B+\alpha(a-b-b\cos A)}{b}.$$

i.e., $$a\alpha(b-c)-b\beta(c-a)\cos C-c\gamma(a-b)\cos B=0,$$

which evidently passes through the point $\frac{1}{b-c},\ \frac{1}{c-a},\ \frac{1}{a-b}$.

Question 431.

(P. V. SESHU AIYAR, B. A., L. T.) :—Integrate

$$\int\frac{\sin\theta}{\sin^{2-\frac{1}{n}}n\theta}\,d\theta$$

Solution by N. Durai Rajan.

Write $n\theta=\phi$ and $\frac{1}{n}=m$.

Then $I=m\int \sin m\phi / \sin^{m-2}\phi.\ d\phi$

$$=m\int \sin^{m-2}\phi(\cos\phi \sin\overline{m-1}.\phi+\sin\phi \cos\overline{m-1}\phi)d\phi.$$

$$=\frac{m}{m-1}.\ \sin^{m-2}\phi.\ \sin(m-1)\phi.$$

Replacing ϕ and n

$$I=\frac{1}{1-n}.\ \sin^{\frac{1}{n}-1} n\theta.\ \sin\left(\frac{1}{n}-1\right)n\theta$$

$$=\frac{1}{1-n}.\ \frac{\sin(1-n)\theta}{\sin^{1-\frac{1}{n}} n\theta}$$

Question 505.

(A. C. L. WILKINSON, M.A., F.R.A.S.) :—A point moves along a curve and PQ is drawn making an angle with the tangent at P equal to λ times the angle which the tangent at P makes with the tangent at a fixed point O of the curve. If PQ envelopes a circle, find the equation of the curve and trace the curves for which $\lambda=1$, $\lambda=\frac{1}{2}$, showing that the latter is a unicursal algebraical curve of the fifth degree whose equation is

$$108\ x^4a+x^2(y^3+162ay^2+405a^2y-2916a^3)+y(y^2+26ay-81a^2)^2=0,$$

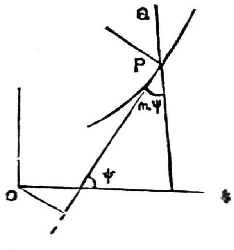
where a is the radius of the given circle, and that the former has an infinite number of parabolic asymptotes given by the equation $y^2=4a+\pi(-x+ar+\pi)$ where r is a positive or negative integer.

Solution by N. Durairajan.

Let

$$x \sin\psi-y\cos\psi=p\equiv f(\psi)$$

be the tangent at P.



Then the normal at P is

$$x\cos\psi+y\sin\psi=\frac{dp}{d\psi}.$$

Since PQ makes an angle $n\psi$ with the tangent at P, its inclination to the x-axis is $(n+1)\psi$. The equation of the PQ is therefore

$$x \sin\psi - y\cos\psi - p + k\left(x\cos\psi + y\sin\psi - \frac{dp}{d\psi}\right) = 0, \quad \ldots \text{(i)}$$

such that the gradient is $\tan(n+1)\psi$.

$$\therefore \quad \frac{\sin\psi + k\cos\psi}{\cos\psi - k\sin\psi} = \tan(n+1)\psi.$$

Substituting in (i) we get

$$x\sin(n+1)\psi - y\cos(n+1)\psi = p\cos n\psi + \frac{dp}{d\psi}\sin n\psi.$$

If this touches a circle of radius a the origin being supposed to be the centre of the circle, we have

$$p\cos n\psi + \frac{dp}{d\psi}\sin n\psi = a.$$

Solving this linear equation, we have

$$p(\sin n\psi)^{\frac{1}{n}} = C + \int a(\sin n\psi)^{\frac{1}{n}-1} d\psi.$$

(i) Let $n=1$, then $p\sin\psi = c + a\psi = a(\psi+\alpha)$, where α is an arbitrary constant.

Taking the typical curve $p = a\psi \operatorname{cosec}\psi$, the tangent and normal are

$$x\sin\psi - y\cos\psi = \frac{a\psi}{\sin\psi},$$

$$x\cos\psi + y\sin\psi = \frac{a\sin\psi - \psi\cos\psi}{\sin^2\psi}.$$

The co-ordinates of P are given by

$$x = a[\cot\psi + \psi(1-\cot^2\psi]$$

$$y = a[1 - 2\psi\cot\psi].$$

The point P moves to infinity where $\psi = r\pi$.

Now, when ψ approaches $r\pi$, we may write

$$\psi = r\pi + t - \frac{t^3}{3} + \frac{t^5}{5} - \ldots\ldots \left[t = \tan\psi\right]$$

Thus

$$\frac{x}{a} = \frac{1}{t} + \left(1 - \frac{1}{t^2}\right)\left(r\pi + t - \frac{t^3}{3} + \ldots\right),$$

$$\frac{y}{a} = 1 - 2\left(r\pi + t - \frac{t^3}{3} + \ldots\right)\frac{1}{t}.$$

$$\therefore \quad \frac{x}{a} = r\pi + t - \frac{t^3}{3} + \ldots - \frac{r\pi}{t^2} - \frac{1}{t} + \frac{t}{3} - \frac{t^3}{5} + \ldots +$$

$$= -\frac{r\pi}{t^2} + r\pi + \frac{4t}{3} - \ldots\ldots$$

$$\frac{y}{a}=1-2\left(\frac{r\pi}{t}+1-\frac{t^2}{3}+\dots\right)$$

$$=-1-\frac{2r\pi}{t}+\frac{2t^2}{3}+\dots\dots$$

If $\psi \to r\pi$, $t \to 0$, and thus

$$\therefore \qquad \frac{x}{a}=-\frac{r\pi}{t^2}+r\pi,$$

$$\frac{y}{a}=-1-\frac{2r\pi}{t}.$$

Eliminating t

$$(y+a)^2=4ar\pi(-x+ar\pi),$$

which is the equation of the parabolic asymptotes.

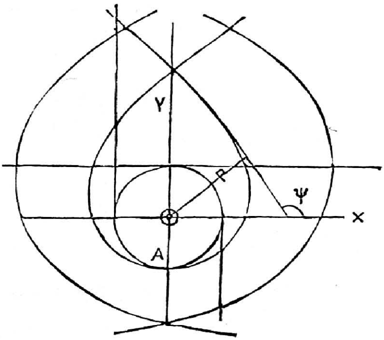
Referring the curve to the point A $(o,-a)$ as origin, the equation becomes $y^2=4ar\pi\ (-x+ar\pi)$.

A tracing of the curve is added. The curve is easily traced by finding out the points where ψ has values

$$o, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi \text{ etc.}$$

The circle is the caustic of the curve for rays parallel to the x—axis

(See Boole's *Differential Equations* for a different method of attacking the problem.)



(ii) Let $n=\frac{1}{2}$.

$$p.\ (\sin \tfrac{1}{2}\psi)^2=C+a\int \sin \tfrac{1}{2}\psi.\ d\psi$$

$$=C-2a \cos \tfrac{1}{2}\psi.$$

Giving C the value $2a$

$$p \sin^2\tfrac{1}{2}\psi=2a(1-\cos \tfrac{1}{2}\psi),$$

24

$$\therefore \qquad p = a \sec^2 \frac{\psi}{4}.$$

Hence $\qquad \frac{x}{a} \cos^2 \tfrac{1}{4}\psi = \sin \psi + \tfrac{1}{2} \cos \psi \tan \tfrac{1}{4}\psi,$

$$\frac{y}{a} \cos^2 \tfrac{1}{4}\psi = -\cos \psi + \tfrac{1}{2} \sin \psi \tan \tfrac{1}{4}\psi.$$

Put $\tan \frac{1}{4}\psi = t$, then x and y can be expressed as functions of a single parameter t.

The eliminant is a unicursal curve. I am not able to find out what value of C Prof. Wilkinson has adopted.

Question 514.

(A. C. L. WILKINSON, M. A., F. R. A. S.) :—All conics cutting a rectangular hyperbola orthogonally at all four points of intersection consist of (1) hyperbolas having the axes of the given hyperbola as asymptotes, (2) all conics confocal with the given hyperbola, (3) two sets of ellipses each passing through two fixed points and whose centres lie on one or other of the axes of the hyperbola and whose axe- are parallel to the asymptotes of the hyperbola.

Solution by N. Durairajan.

A more general problem is "given a conic, find all the conics cutting the given conic orthogonally at all four points of intersection."

Let the given conic be

$$S \equiv ax^2 + by^2 - 1 = 0,$$

and the orthogonal conic be

$$S' \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0.$$

The condition for orthogonality is

$$\phi \equiv ax(a'x + h'y + g') + by(h'x + b'y + f') = 0.$$

This is true for the four points of intersection.

The conic ϕ is therefore of the form $\lambda S + \mu S' = 0$.

Equating co-efficients

$$\begin{aligned} aa' &= \lambda a + \mu a' \\ bb' &= \lambda b + \mu b' \\ h'(a+b) &= 2\mu h' \\ ag' &= 2\mu g' \\ bf' &= 2\mu f' \\ 0 &= -\lambda + \mu c'. \end{aligned}$$

The third condition gives

$$h' = 0, \text{ or } a + b = 2\mu.$$

The fourth and fifth conditions given

$$g'=0, \text{ or } a=2\mu$$
$$f'=0, \text{ or } b=2\mu.$$

We have thus the following cases :—

(1) $h'=0,\ g'=0,\ f'=0.$

$$\text{S}' \text{ becomes } \frac{\lambda a}{a-\mu}\cdot x^2+\frac{\lambda b}{b-\mu}\cdot y^2+\frac{\lambda}{\mu}=0.$$

$$\therefore \qquad \frac{a}{a-\mu}x^2+\frac{b}{b-\mu}y^2+\frac{1}{\mu}=0.$$

These are conics confocal with $ax^2+by^2=1$.

(2) $h\neq 0$, *i.e.*, $\mu=\frac{1}{2}(a+b),\ g'=0, f'=0.$

$$\therefore \qquad a'=\frac{\lambda a}{a-\mu}=\frac{2\lambda a}{a-b},\ b'=\frac{2\lambda b}{b-a}$$

Hence S′ becomes

$$2h'xy+\frac{2\lambda}{a-b}\left(ax^2-by^2+\frac{a-b}{a+b}\right)=0,$$

shewing that the conics pass through the intersections of

$$xy=0, \text{ and } ax^2-by^2+\frac{a-b}{a+b}=0.$$

(3) $h'=0,\ g'=0,\ b=2\mu.$

$$\therefore \qquad a'=\frac{2\lambda a}{2a-b},\ b'=2\lambda,\ c'=\frac{2\lambda}{b}.$$

Thus $\text{S}'\equiv 2\lambda\left(\frac{ax^2}{2a-b}+y^2+\frac{1}{b}\right)+2f'y=0.$

(4) $h'=0,\ a=2\mu,\ f'=0$

Similarly, we get

$$x^2+\frac{by^2}{2b-a}+\frac{1}{a}=\lambda x.$$

(5) $h'=0,\ a=2\mu,\ b=2\mu.$

This case is impossible, for, at the points of intersection of the given circle $(a=b)$, the normals to the circle are tangents to the conic: The conic would have four tangents from the centre of the circle, which is not possible.

(6) $h'\neq 0,\ g'=0,\ b=2\mu$, and $2\mu=a+b.$

Here $a=0.$, which is contrary to the hypothesis.

So also $h'\neq 0,\ f'=0,\ a=2\mu.$

The condition $\phi=0$ may be identically satisfied by putting.

$$aa'=0,\ bb'=0,\ h'(a+b)=0,\ ag'=0,\ bf'=0.$$

Since a and b are not zero, $a'=0,\ b'=0$, and h' **is not** zero, S′ being a conic..

$\therefore \qquad a+b=0$ and $g'=0, f'=0.$

N. B.—When the given conic is a R. H. $x^2-y^2-\frac{1}{a}=0$, the orthogonal conic is the R.H. $xy=\beta$.

Question 521.

(K. V. Anantanarayana Sastry, B.A.) :—If V denote the entire volume of the figure bounded by

$$\left(\frac{x}{a}\right)^{\frac{16}{17}}+\left(\frac{y}{b}\right)^{\frac{16}{17}}+\left(\frac{z}{c}\right)^{\frac{16}{17}}=1,$$

and A the whole area of its trace on the xy—plane, prove that

$$V=\frac{17.Ac}{19.35}. \text{ B}\left(\frac{1}{8}, \frac{1}{16}\right).$$

Solution by T. P. Trivedi, M.A., LL.B., R. J. Pocock and P. A. Subramanya Iyer B.A., L.T.

The volume is given by $8\int\int\int dx\ dy\ dz$ where the integral is to extend to all *positive* values of x, y, z subject to the condition

$$\left(\frac{x}{a}\right)^{\frac{16}{17}}+\left(\frac{y}{b}\right)^{\frac{16}{17}}+\left(\frac{z}{c}\right)^{\frac{16}{17}}=1.$$

Employing Dirichlet's formula, the volume

$$V=\frac{8\ abc}{\left(\frac{16}{17}\right)^3}\ \frac{\Gamma\left(\frac{17}{16}\right)^3}{\Gamma\left(\frac{51}{16}+1\right)}.$$

The area of the trace on the xy-plane is

$$\frac{4\ ab}{\left(\frac{16}{17}\right)^2}\ \frac{\Gamma\left(\frac{17}{16}\right)^2}{\Gamma\left(\frac{34}{16}+1\right)}.$$

$$\therefore \qquad \frac{V}{A}=2c.\frac{17}{16}\ \frac{\Gamma\left(\frac{17}{16}\right)\Gamma\left(\frac{50}{16}\right)}{\Gamma\left(\frac{67}{16}\right)}$$

$$=2c.\frac{\frac{17}{16}.\frac{1}{16}.\frac{34}{16}.\frac{18}{16}.\frac{2}{16}}{\frac{51}{15}.\frac{35}{16}.\frac{19}{16}.\frac{3}{16}}.\frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{1}{16}\right)}{\Gamma\left(\frac{3}{16}\right)}$$

$$=\frac{17c}{19.35}\text{ B}\left(\frac{1}{8}, \frac{1}{16}\right).$$

Question 523.

(R. N. APTE M.A., F.R.A.S.) :—Find the value of

$$\int\int xyz\left\{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}dx\,dy,$$

where $z=c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}$, and the integration is over the positive quadrant of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution by T. P. Trivedi M.A., LL.B.

Since $$\frac{z}{c}=\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}},\ \frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}.$$

$$\frac{z}{c^2}dz=-\frac{x}{a^2}dx,\text{ or }\ \frac{dz}{dx}=-\frac{x\,c^2}{z\,a^2};$$

Similarly $$\frac{dz}{dy}=-\frac{y\,c^2}{z\,b^2}.$$

Thus the integral

$$=\int\int\frac{xy}{z}\left\{z^2+\frac{c^4}{a^4}x^2+\frac{c^4}{b^4}y^2\right\}dxdy.$$

$$=\int\int\frac{xy}{c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}}\left\{c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+\frac{c^4}{a^4}x^2+\frac{c^4}{b^4}y^2\right\}dxdy$$

Change x into ax and y into by ; then $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ becomes x^2+y^2 and $dx\,dy$ becomes $abdxdy$. The integral finally reduces to

$$a^2b^2c\int_0^1\int_0^{\sqrt{1-x^2}}\frac{x}{(1-x^2-y^2)^{\frac{1}{2}}}\left\{(1-x^2-y^2)+\frac{c^2}{a^2}x^2+\frac{c^2}{b^2}y^2\right\}dx\,dy$$

$$=a^2b^2c\int_0^1\left[\int_0^{\sqrt{1-x^2}}\frac{xy}{(1-x^2-y^2)^{\frac{1}{2}}}\left(1-x^2+\frac{c^2}{a^2}x^2\right)dy\right.$$
$$\left.+\int_0^{\sqrt{1-x^2}}\frac{xy^3}{(1-x^2-y^2)^{\frac{1}{2}}}\left(\frac{c^2}{b^2}-1\right)dy\right]dx$$

$$=a^2b^2c\int_0^1\left[-x(1-x^2-y^2)^{\frac{1}{2}}\left((1-x^2+\frac{c^2}{a^2}x^2\right)+\frac{1}{3}x(1-x^2-y^2)^{\frac{3}{2}}\right.$$
$$\left.\times\left(\frac{c^2}{b^2}-1\right)-x(1-x^2)(1-x^2-y^2)^{\frac{1}{2}}\left(\frac{c^2}{b^2}-1\right)\right]_0^{\sqrt{1-x^2}}dx$$

$$=a^2b^2c\int_0^1\left\{x\sqrt{1-x^2}\left(1-x^2+\frac{c^2}{a^2}x^2\right)+\frac{2}{3}x\left(\frac{c^2}{b^2}-1\right)(1-x^2)^{\frac{3}{2}}\right\}dx.$$

Put $x=\sin\theta$;

$$I=a^2b^2c\int_0^{\frac{\pi}{2}}\left\{\sin\theta\cos^2\theta\left(\cos^2\theta+a^2\sin^2\theta\right)+\frac{2}{3}\cos^4\theta\sin\theta\left(\frac{c^2}{b^2}-1\right)d\theta\right\}$$

which gives on reduction

$$\frac{a^2b^2c^3}{15}\left(\frac{2}{a^2}+\frac{2}{b^2}+\frac{1}{c^2}\right).$$

Question 524.

(S. Ramanujan) :—Shew that

$$\text{(i)}\quad \sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{\pi}{7}}+\sqrt{\cos\frac{8\pi}{7}}=\sqrt[3]{\frac{5-3\sqrt[6]{7}}{2}}$$

$$\text{(ii)}\quad \sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt{\cos\frac{8\pi}{9}}=\sqrt[3]{\frac{3\sqrt[3]{9}-6}{2}}.$$

Solution by N. Sankara Aiyar, M. A.

Lemma.—If α,β,γ, be the roots of a given cubic equation to form the equation satisfied by p where $\sqrt[3]{p}=\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}$.

Cubing we get

$$p=\Sigma\alpha+3(\Sigma\sqrt[3]{\alpha})(\Sigma\sqrt[3]{\alpha\beta})-3\sqrt[3]{\alpha\beta\gamma}.$$

$$\therefore\ \{p-\Sigma\alpha+3\sqrt[3]{\alpha\beta\gamma}\}^3=27p\{\Sigma\alpha\beta+3\Sigma(\sqrt[3]{\alpha\beta})\Sigma(\sqrt[3]{\alpha^2\beta\gamma})-3\sqrt[3]{\alpha^2\beta^2\gamma^2}\}$$

$$=27p\{\Sigma\alpha\beta+\sqrt[3]{\alpha\beta\gamma}(p-\Sigma\alpha+3\sqrt[3]{\alpha\beta\gamma})-3\sqrt[3]{\alpha^2\beta^2\gamma^2}\}$$

$$\therefore\ p^3-3p^2(\Sigma\alpha-3\sqrt[3]{\alpha\beta\gamma})+3p(\Sigma\alpha-3\sqrt[3]{\alpha\beta\gamma})^2$$
$$-(\Sigma\alpha-3\sqrt[3]{\alpha\beta\gamma})^3=27p\{\Sigma\alpha\beta-\Sigma\alpha\sqrt[3]{\alpha\beta\gamma}\}+27p^2\sqrt[3]{\alpha\beta\gamma}.$$

$$i.e.\quad p^3-3p^2(\Sigma\alpha+6\sqrt[3]{\alpha\beta\gamma})+3p\{(\Sigma\alpha-3\sqrt[3]{\alpha\beta\gamma})^2-9\Sigma\alpha\beta$$
$$+9\Sigma\alpha\sqrt[3]{\alpha\beta\gamma}\}-(\Sigma\alpha-3\sqrt[3]{\alpha\beta\gamma})^3=0.$$

(i) Let $x=\cos\frac{2r\pi}{7}+i\sin\frac{2r\pi}{7}$. The roots of $x^7-1=0$ are the values of $\cos\frac{2r\pi}{7}+i\sin\frac{2r\pi}{7}$.

Substitute $x+\frac{1}{x}=y=2\cos\frac{2\gamma\pi}{7}$. The equation for y is

$y^3+y^2-2y-1=0$, and its roots are

$$2\cos\frac{2\pi}{7},\ 2\cos\frac{4\pi}{7},\ 2\cos\frac{6\pi}{7}\ \text{or}\ 2\cos\frac{8\pi}{7}.$$

Substituting in the equation for p the values of $\Sigma\alpha$ $\Sigma\alpha\beta$, and $\alpha\beta\gamma$, we get after reduction

$$p^3-15p^2+75p+64=0,$$

or $$(p-5)^3+189=0.$$

The real value of p is thus $(5-3\sqrt[3]{7})$.

Hence

$$\sqrt[3]{2\cos\frac{2\pi}{7}}+\sqrt[3]{2\cos\frac{4\pi}{7}}+\sqrt[3]{2\cos\frac{8\pi}{7}}=\sqrt[3]{5-3\sqrt[3]{7}}.$$

$$\therefore\quad \sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}=\sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}.$$

(ii) Again the equation

$$(x^9-1)/(x^3-1)=0,$$

has roots $\cos\frac{2r\pi}{9}+i\sin\frac{2r\pi}{9}$, where $r \not\equiv 0 \pmod 3$.

Substituting $x+\frac{1}{x}=y=2\cos\frac{2r\pi}{9}$, we get the equation for y in the form $y^3-3y+1=0$.

The equation for p reduces as before to

$$p^3+18p^2+108p-27=0,$$

or $$(p+6)^3-248=0.$$

$$\therefore\quad p=3\sqrt[3]{9}-6.$$

Thus

$$\sqrt[3]{2\cos\frac{2\pi}{9}}+\sqrt[3]{2\cos\frac{4\pi}{9}}+\sqrt[3]{2\cos\frac{8\pi}{9}}=\sqrt[3]{3\sqrt[3]{9}-6}$$

$$\therefore\quad \sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{6}}+\sqrt[3]{\cos\frac{8\pi}{9}}=\sqrt[3]{\frac{3\sqrt[3]{9}-6}{2}}.$$

Question 525.

(S. Ramanujan) :—Show how to find the square roots of surds of the form $\sqrt[3]{A}+\sqrt[3]{B}$, and hence prove that

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}=\frac{\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}}{3}$$

$$\sqrt{\sqrt[3]{28}-\sqrt[3]{27}}=\frac{\sqrt[3]{98}-\sqrt[3]{28}-1}{3}.$$

Solution by N. Sankara Aiyar, M.A.

The square roots should evidently be of the form $l\sqrt[3]{p}+m\sqrt[3]{q}+n\sqrt[3]{r}$, where the surds $\sqrt[3]{p^2}$, $\sqrt[3]{pq}$ &c., should be similar in pairs

$\therefore\quad p=a^2b,\ q=b^2,\quad r=a$ ... ... (i)

or $\quad p=ab^2,\ q=b^3,\quad r=a^2b^2$ ... ... (ii)

or $\quad p=a^3,\quad q=ab^2,\quad r=a^2b$ ... ... (iii)

Taking (i) we get the square

$$\sqrt[3]{ab^2}(al^2+2mn)+\sqrt[3]{b}(bm^2+2aln)+\sqrt[3]{a^2}(n^2+2\,lmb).$$

One of these coefficients should vanish.

Thus, taking the first example, we get

$$al^2+2mn=0,\ bm^2+2aln=1,$$
$$n^2+2blm=-1 \text{ and } b=5,\ a=2.$$

Hence $$l=\frac{1}{3},\ m=-\frac{1}{3},\ n=\frac{1}{3}.$$

$$\therefore \quad \sqrt{\sqrt[3]{5}-\sqrt[3]{4}}=\frac{\sqrt[3]{20}-\sqrt[3]{25}+\sqrt[3]{2}}{3}.$$

Taking (ii) we get the sqnare in the form

$$\sqrt[3]{a^2b}(bl^2+2bmn)+\sqrt[3]{b}(bm^2+2abln)+\sqrt[3]{ab}(abn^2+2blm).$$

We should make one of these coefficients vanish.

Taking (iii) we get the square in the form

$$l^2a^2+2mnab+\sqrt[3]{ab^2}(an^2+2alm)+\sqrt[3]{a^2b}(bm^2+2aln).$$

Comparing this with the second example above, we get,

$$l^2a^2+2mnab=-3,\ bm^2+2aln=0,$$
$$an^2+2alm=1,\ a=7,\ b=2;$$

or

$$l^2a^2+2mnab=-3,\ bm^2+2aln=1,$$
$$an^2+2alm=0,\ a=2,\ b=7.$$

The first gives $l=\frac{1}{21},\ m=\frac{7}{21},\ n=-\frac{7}{21},$

The second gives $l=\frac{1}{6},\ m=-\frac{1}{3},\ n=\frac{1}{3}.$

Both give the same result : *viz.*,

$$\frac{1+\sqrt[3]{28}-\sqrt[3]{98}}{3}.$$

The positive root is therefore

$$\frac{\sqrt[3]{98}-\sqrt[3]{28}-1}{3}.$$

Question 527.

(A. Narasinga Rao.) :—A family of quartics having nodes at A, B, C, pass through four other given points. Shew that only two members of the family touch the circle ABC.

Solution by Proposer.

The isogonal transformation of this quartic family *w.r.t.* Δ ABC is a family of conics passing through the conjugates of the four given points. Also, the isogonal transform of the circle ABC is the line at infinity. Hence the above result follows from the following property of conics :

"Two members of a four point system of conics touch the line at infinity."

Question 528.

(A. N. RAGHAVACHAR, M. A.) :—If (n, r) denote the sum of the products r together of the first n natural numbers, find the value of

$$\frac{(2n-1,n)}{\underline{|2n}}-\frac{(2n-2,n)}{\underline{|1}\,\underline{|2n-1}}+\frac{(2n-3,n)}{\underline{|2}\,\underline{|2n-2}}+\cdots\frac{(-1)^{n-1}(n,n)}{\underline{|n-1}\,\underline{|n+1}}.$$

Solution by T. P. Trivedi, M.A., LL. B.

We have

$$\frac{[\log(1+x)]^r}{\underline{|r}}=\frac{x^r}{\underline{|r}}-\frac{(r,1)x^{r+1}}{\underline{|r+1}}+\frac{(r+1,2)x^{r+2}}{\underline{|r+2}}-\cdots$$

(Ex. 11, page 80, Edward's *Diff. Calc.*)

Hence

$$\frac{(2n-1,n)}{\underline{|2n}}=\text{coefficient of } x^{2n} \text{ in } (-1)^n\frac{[\log(1+x)]^n}{\underline{|n}},$$

$$\frac{(2n-2,n)}{\underline{|1}\,\underline{|2n-1}}=\text{coefficient of } x^{2n} \text{ in } (-1)^n\frac{x[\log(1+x)]^{n-1}}{\underline{|1}\,\underline{|n-1}},$$

and so on. Thus the series is equal to the coeff. of x^{2n} in

$$(-1)^n\frac{1}{\underline{|n}}\Big\{[\log(1+x)]^n-\frac{\underline{|n}}{\underline{|1}\,\underline{|n-1}}x[\log(1+x)]^{n-1}$$

$$+\frac{\underline{|n}}{\underline{|n-2}}x^2[\log(1+x)]^{n-2}-\text{etc. to } n \text{ terms}\Big\}$$

$$=(-1)^n \text{ coeff. of } x^{2n} \text{ in } \frac{1}{\underline{|n}}\Big\{[\log(1+x)-x]^n+(-1)^{n-1}x^n\Big\}$$

$$=(-1)^n \text{ coeff. of } x^{2n} \text{ in } \frac{1}{\underline{|n}}\Big\{\Big(-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\Big)^n+(-1)^{n-1}x^n\Big\}$$

$$=\frac{1}{\underline{|n}\,2^n}.$$

Question 531.

(N. SANKARA AIYAR, M.A.) :—Integrate

$$\int_0^\infty\frac{\sin xy}{x(1+x^2)^3}dx.$$

Solutions : (1) by J. M. Bose, (2) by K. J. Sanjana, T. P. Trivedi and P. A. Subra- manya Iyer, B.A., L.T.; (3) by the Proposer.

(1) Let $\mathrm{I}=\int_{-\infty}^{\infty}\frac{\sin xy}{x(1+x^2)^3}$, so that $\frac{\partial \mathrm{I}}{\partial y}=\int_{-\infty}^{\infty}\frac{\cos xy}{(1+x^2)^3}$.

Now $\frac{\partial I}{\partial y}$ is the real part of $\frac{e^{yzi}}{(1+z^2)^3}$ taken round a suitable contour in the z-plane.

Consider a contour composed of a semicircle in the upper half of the z-plane together with the real axis from $-\infty$ to $+\infty$.

The function has only one singularity $z=i$ of the third order within the contour.

Let R be the residue at $z=i$.

Then $$R = \underset{z=i}{L}\ \frac{1}{\underline{|2}} \left(\frac{d}{dz}\right)^2 \left[\frac{e^{yzi}}{(1+z^2)^3}(z-i)^3\right];$$

on differentiating and putting $z=i$, we have

$$R = \frac{-ie^{-y}}{16}(y^2+3y+3),$$

$$\frac{\partial I}{\partial y} = 2\pi i R = \frac{\pi}{8} e^{-y}(y^2+3y+3).$$

$\therefore$ $$I = \frac{\pi}{8}\int e^{-y}(y^2+3y+3)dy + \text{const.}$$

$$= \frac{\pi}{8}\left[8 - e^{-y}(y^2+5y+8).\right]$$

Since the integrand is even the reqd. integral $= \frac{1}{2}$I.

Hence $$\int_0^\infty \frac{\sin xy}{x(1+x^2)^3}dx = \frac{\pi}{16}\left[8 - e^{-y}(y^2+5y+8).\right]$$

(2) Taking the result

$$\int_0^\infty \frac{\cos xy}{(1+x^2)^3}dx = \frac{\pi}{16}e^{-y}(3+3y+y^2),$$ [*Journal*, Dec. 1913, p. 231]

integrating with respect to y, and determining the constant so that the integral vanishes with y, we obtain

$$\int_0^\infty \frac{\sin xy}{x(1+x^2)^3}dx = C + \frac{\pi}{16}\{-3e^{-y} - (3e^{-y}+ye^{-y})$$
$$-(2e^{-y}+2ye^{-y}+y^2e^{-y})\}$$
$$= \frac{\pi}{16}\left\{8 - e^{-y}(8+5y+y^2)\right\}.$$

(3) Let J = the required value of the integral. Then

$$\frac{d^2J}{dy^2} - J = -\int_0^\infty \frac{\sin xy}{x(1+x^2)^2}dx$$
$$= \frac{\pi}{4}ye^{-y} + \frac{\pi}{2}e^{-y} - \frac{\pi}{2}$$

(Vol. VI. p. 26 of J. I. M. S.)

$\therefore$ $$J = Ae^{y} + Be^{-y} - \frac{\pi}{16}(y^2+5y)e^{-y} + \frac{\pi}{2}.$$

But J = 0 when $y=0$, and when $y=0$

$$\frac{dJ}{dy}=\int_0^\infty \frac{\cos xy}{(1+x^2)^3}dx=\int_0^\infty \frac{dx}{(1+x^2)^3}=\frac{3\pi}{16}.$$

$\therefore$
$$A+B+\frac{\pi}{2}=0,$$

and
$$A-B-\frac{5\pi}{16}=\frac{3\pi}{16},$$

$\therefore$
$$A=0,\ B=-\frac{\pi}{2}.$$

Hence
$$J=\frac{\pi}{2}-e^{-y}(y^2+5y+8)\frac{\pi}{16}$$

Question 536.

(K. Appukuttan Erady M. A.) :—Prove that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ue^{-u'}\,dxdy=\frac{\pi}{2}\cdot\frac{ab'+a'b-2hh'}{(a'b'-h'^2)^{\frac{3}{2}}},$$

where $u\equiv ax^2+2hxy+by^2$; $u\equiv a'x^2+2h'xy+b'y^2$.

Solution (1) *by R. Vythynathaswamy and* (2) *by R. J. Pocock, B.A., B.Sc , F.R.A.S.*

(1) Consider the pairs of lines represented by

$ax^2+2h\ xy+by^2=0$ ... ... (1)

$a'x^2+2h'xy+b'y^2=0.$ ... ... (2)

Transform the axes so as to coincide with the bisectors of the angles between the lines (2). Let the transformed equations be

$Ax^2+2H\ xy+By^2=0$ ... ... (1')

$Lx^2+My^2=0$ ... ... (2')

Then the given integral reduces to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(Ax^2+2H\,xy+By^2\right)e^{-Lx^2,-My^2}\,dx\,dy,$$

since
$$\int_{-\infty}^{\infty} xe^{-Kx^2}dx=0.$$

$$\text{The above}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(Ax^2+By^2)\,e^{-L.x^2}e^{-My^2}dxdy.$$

Noting that, $\int_{-\infty}^{\infty} e^{-ax^2}dx=2\int_{-0}^{\infty} e^{-ax^2}dx$, the above

$$=\int_{-\infty}^{\infty}\left(\frac{A}{2}\frac{\sqrt{\pi}}{L^{\frac{3}{2}}}e^{-My^2}+\frac{\sqrt{\pi}}{L}By^2.\,e^{-My^2}\right)dy$$

$$=\sqrt{\frac{\pi}{M}}\cdot\frac{A}{2}\cdot\frac{\sqrt{\pi}}{L^{\frac{3}{2}}}+\sqrt{\frac{\pi}{L}}\cdot\frac{\sqrt{\pi}}{M^{\frac{3}{2}}}\cdot\frac{B}{2}$$

$$=\frac{\pi}{2}\left(\frac{AM+BL}{(LM)^{\frac{3}{2}}}\right)$$

And by invariants

$$AM+BL=ab'+a'b-2hh',$$

and

$$LM=a'b'-h'^2,$$

whence the result reduces to

$$\frac{\pi}{2}\,\frac{(ab'+a'b-2hh')}{(a'b'-h'^2)^{\frac{3}{2}}}.$$

(2) By the substitution $X\sqrt{a'}=a'x+h'y$; $Y\sqrt{a'}=y\sqrt{a'b'-h'^2}$, u' becomes X^2+Y^2, and u becomes

$$\frac{a}{a'}\cdot X^2+\frac{2X\cdot Ya}{\sqrt{a'b'-h'^2}}\cdot\left(h-\frac{h'}{a'}\right)+\frac{Y^2}{(a'b'-h'^2)}\cdot\left\{\frac{ah'^2}{a'}-2\,hh+a'b\right\}.$$

Also $dxdy$ becomes $\frac{dXdY}{\sqrt{a'b'-h'^2}}$ (Williamson's *Integral*, § 223).

Now
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+}XYe^{-X^2-Y^2}dX\,dY=$$

$$=\int_{-\infty}^{+\infty}Xe^{-X^2}dX\int_{-\infty}^{+\infty}Ye^{-Y^2}dY=0,$$

since the integrals vanish at both limits.

Also

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}X^2e^{-X^2-Y^2}dX\,dY=\int_{-\infty}^{+\infty}X^2e^{-X^2}dX=\int_{-\infty}^{+\infty}e^{-Y^2}dY$$

$$=4\int_{0}^{+\infty}X^2e^{-X^2}\,X\int_{0}^{+\infty}e^{-Y^2}dY,$$

$$=2\int_{0}^{+\infty}z^{\frac{1}{2}}e^{-z}dz\int_{0}^{+\infty}e^{-Y^2}dY,$$

writing $X^2=z$.

The first integral$=\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2}$, the second also$=\frac{\sqrt{\pi}}{2}$.

The given integral

$$=\frac{1}{\sqrt{a'b'-h'^2}}\left\{\frac{a}{a'}+\left(\frac{ah'^2}{a}-2hh'+a'b\right)\frac{1}{(a'b'-h'^2)}\right\}\frac{\pi}{2},$$

$$=\frac{\pi}{2}\frac{ab'+a'b-2hh'}{(a'b'-h'^2)^{\frac{3}{2}}}.$$

Question 537.

(K. J. SANJANA, M.A.) :—Prove that

$$\frac{1}{2}\cdot\frac{x^2}{2}-\frac{1.3}{2.4}\frac{x^4}{4}+\frac{1.3.5}{2.4.6}\frac{x^6}{6}-\ldots\ldots$$

$$=\frac{1}{4}\log(1+x^2)+\frac{1}{8}\log(1+x_1^2)+\frac{1}{16}\log(1+x_2^2)+\ldots\ldots$$

where $x_1^2=\dfrac{x^4}{4(x^2+1)}$, $x_2^2=\dfrac{x_1^4}{4(x_1^2+1)}\ldots\ldots;$

and that

$$\frac{1}{2}\frac{x^2}{2}+\frac{1.3}{2.4}\frac{x^4}{4}+\frac{1.3.5}{2.4.6}\frac{x^6}{6}+\ldots$$

$$=-\tfrac{1}{4}\log(1-x^2)-\tfrac{1}{8}\log(1+x_1^2)-\tfrac{1}{16}\log(1+x_2^2)\ldots$$

where $x_1^2=\dfrac{x^4}{4(1-x^2)}$, $x_2^2=\dfrac{x_1^4}{4(1+x_1^2)},\ldots\ldots$

In the first equality x is any real quantity ; in the second, any real quantity <1.

Solution by T. P. Trivedi, M.A., L L.B.

We have

$$\int_0^{\frac{\pi}{2}}\log(1+n\cos^2\theta)d\theta=\frac{\pi}{4}\log\left\{(1+n)(1+n_1)^{\frac{1}{2}}(1+n_2)^{\frac{1}{4}}\ldots\ldots\right\},$$

where n, n_1, n_2 etc., are connected by the relation $n_{r+1}=\dfrac{n_r}{4(n_r+1)}$.

(Ex 15, page 75, Todhunter's *Integ. Calc.*)

Taking $n=x^2$, we find that

$$\tfrac{1}{4}\log(1+x^2)+\tfrac{1}{8}\log(1+x_1^2)+\ldots\ldots=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\log(1+x^2\cos^2\theta)d\theta$$

$$=\frac{1}{\pi}\left\{\int_0^{\frac{1}{2}\pi}(x^2\cos^2\theta-\frac{x^4\cos^4\theta}{2}+\ldots\ldots)d\theta\right.$$

$$=\frac{1}{2}\cdot\frac{x^2}{2}-\frac{1.3}{2.4x^4}+\frac{1.3.5}{2.4.6}\frac{x^6}{6}-\ldots\ldots$$

= the left side of the first equality.

The second result follows by changing x into xi or by reducing $\int_0^{\frac{1}{2}\pi}\log(1-x^2\cos^2\theta)d\theta$, as in the first case.

QUESTIONS FOR SOLUTION.

575. (S. NARAYANA AIYAR, M.A.) :—If B_1, B_3, B_5, B_7 etc., are the Bernoulli's numbers $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}$, etc., show that

$$B_1 - B_3 + B_5 - B_7 + B_9 - B_{11} + \ldots\ldots \text{to } \infty = \frac{\pi^2 - 9}{6}.$$

576. (S. NARAYANA AIYAR, M.A.) :—Shew that

$$e^x = \frac{1 + x\cdot\frac{a}{b} + \frac{x^2}{1\cdot2}\cdot\frac{a(a+1)}{b(b+1)} + \ldots\ldots\ldots \text{to } \infty}{1 + x\cdot\frac{a-b}{b} + \frac{x^2}{1\cdot2}\cdot\frac{(a-b)(a-b-1)}{b(b+1)} + \ldots\ldots\ldots \text{to } \infty}$$

577. (K. APPUKUTTAN ERADY, M.A.) :—If S, S′ are the semi-sums of opposite angles of a spherical quadrilateral, prove that

$$\cos(S-S')\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c\sin\tfrac{1}{2}d - \cos(S+S')\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\tfrac{1}{2}d = \tfrac{1}{4}\Sigma(\cos a),$$

a, b, c, d being the sides of the quadrilateral.

578. (N. P. PANDYA) :—Construct a triangle ABC, having given the side BC, the distance between the circumcentre and the symmedian point, and also the distance between the incentre and the excentre.

579. (S. NARAYANAN, B.A., L.T.) :—With the three vertices of a triangle ABC as centres, circles are described to touch the opposite sides. Shew that the three common chords of these circles with the circumcircle from a triangle which is in perspective with the original iangle.

580. (R. VYTHYNATHASWAMY) :—Shew that the algebraical factors of the determinant

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & l^2 \\ 1 & c^2 & 0 & a^2 & m^2 \\ 1 & b^2 & a^2 & 0 & n^2 \\ 1 & l^2 & m^2 & n^2 & 0 \end{vmatrix}$$

are given by

$$\{ \pm f(a, m, n) \pm f(b, n, l) \pm f(c, l, m) - f(a, b, c) \},$$

the ambiguous signs being taken all positive or two positive, where

$$\{f(x,y,z)\}^2 = \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}$$

581. (R. VYTHYNATHASAWMY) :—Shew that

(1) ${}_1C_0 - {}_5C_2 + {}_9C_4 - {}_{13}C_6 \ldots\ldots$ to $\infty = \frac{1}{2}\sin(\frac{1}{2}\tan^{-1}4)$,

(2) $-{}_3C_2 + {}_7C_4 - {}_{10}C_6 \ldots\ldots$ to $\infty = \frac{1}{2}\cos(\frac{1}{2}\tan^{-1}4)$.

582. (J. C. SWAMINARAYAN, M.A.) :—Prove the identities

$$\tan^{-1}x - \tan^{-1}\frac{x}{3} + \tan^{-1}\frac{x}{5} - \tan^{-1}\frac{x}{7} + \ldots\ldots = \tfrac{1}{2}\,\mathrm{gd}\,\frac{\pi}{2}x,$$

and

$$\tanh^{-1}x - \tanh^{-1}\frac{x}{3} + \tanh^{-1}\frac{x}{5} - \tanh^{-1}\frac{x}{7} + \ldots\ldots\ldots = \tfrac{1}{3}\,\mathrm{gd}^{-1}\left(\frac{\pi}{2}x\right).$$

583.—(J. C. SWAMINARAYAN, M.A.) :—Shew that

$$\int_0^{\frac{\sqrt{5}-1}{2}} \frac{\log(1+x)}{x}dx = \frac{\pi^2}{15} - \tfrac{1}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

and

$$\int_0^{\frac{\sqrt{5}-1}{2}} \frac{\log(1-x)}{x}dx = \left\{\log\frac{\sqrt{5}-1}{2}\right\}^2 - \frac{\pi^2}{10}.$$

(Suggested by Question 572 of Mr. Bhimansena Rao.)

584. (S. RAMANUJAN) :—Examine the correctness of the following results :—

(i) $$1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \ldots\ldots$$
$$= \frac{1}{(1-x)(1-x^6)(1-x^{11})(1-x^{16})\ldots} \times \frac{1}{(1-x^4)(1-x^9)(1-x^{14})\ldots\ldots}$$

Here 1, 4, 9...on the left side are the squares of natural numbers. while 1, 6, 11, 16...and 4, 9, 14,...on the right side are numbers in A.P. with 5 for common difference.

(ii) $$1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \ldots\ldots$$
$$= \frac{1}{(1-x^2)(1-x^7)(1-x^{12})\ldots} \times \frac{1}{(1-x^3)(1-x^8)(1-x^{13})\ldots\ldots}$$

Here the nth term of the series 2, 6, 12...is $n\ (n+1)$, and 2, 7, 12...; 3, 8, 13,...increase by 5.

585. (K. V. ANANTHANARAYANA SASTRI, B.A.) :—Prove that

$$\int_0^{\frac{\pi}{2}} \tan^{-1} (a \sin \phi)\, d\phi = \frac{\pi^2}{4} + \log m \log\left(\frac{1-m}{1+m}\right) - 2\sum_0^{\infty} \frac{m^{2r+1}}{(2r+1)^2},$$

where $m=(1+a^2)^{\frac{1}{2}}-a$.

586. (K. V. ANANTANARAYANA SASTRI, B.A.) :—Prove that the length of the first negative pedal with respect to the origin of the loop of the Folium of Descartes $x^3+y^3-3\,axy=0$ is equal to $6a-a\left\{\pi-\sqrt{2}\log(1+\sqrt{2})\right\}$.

587. (T. P. TRIVEDI, M. A., LL.B.) :—If the centre of pressure of a triangle immersed in a liquid coincides with the symmedian point prove that the depths of the angular points are in the ratios

$$3a^2-b^2-c^2\ ;\ 3b^2-c^2-a^2;\ 3c^2-a^2-b^2.$$

588. (T. P. TRIVEDI, M.A., LL.B.) :—If S_n denote the series

$1-\frac{1}{3^3}+\frac{1}{5^3}-$etc., to n terms, prove that

$$\sum_{n=1}^{n=\infty} \frac{(-1)^{n-1} S_n}{(2n+1)^3} = \frac{\pi^6}{15\cdot 2^{11}}$$

589. (S. KRISHNASAWMI AIYANGAR) :— ABCD is a quadrilateral and its diagonals meet at right angles at S. With S as focus two conics α, β are described one circumscribing the quadrilateral and the other inscribed in it. Show that

1. the two conics have one directrix identical,
2. the envelope of polars with respect to β of points on α is a conic having one focus and one directrix common with those of α and β.
3. the lines joining the points of contact of β with the sides of the quadrilateral meet two by two at the vertices of the harmonic triangle of the quadrilateral.

590. (A. A. KRISHNAWAMI AIYANGAR) :—Solve completely :

$$(1-4x^2)^2\frac{d^2y}{dx^2}+(1-4x^2)\frac{dy}{dx}+(2x+1)\ (2x+3)y=0.$$

591. (A. A. KRISHNASWAMI AIYANGAR) :—Two circles intersect at A and B. The tangents at the extremities of a double chord through A meet in X. Show that XY perpendicular to BX envelops a circle.